Journal of Mathematical Chemistry 12(1993)173–195

Enumeration of digraphs with a given automorphism group*

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Four methods are described for enumerating digraphs with a given automorphism group: (1) a generating-function method based on subduced cycle indices, (2) a generating-function method based on partial cycle indices, (3) a method based on the elementary superposition theorem, and (4) a method based on the partial superposition theorem. All of these methods are based on the concept of unit subduced cycle indices and construct a set of versatile tools for combinatorial enumeration. They are applied to the enumeration of five-vertex digraphs with a given automorphism group. The table of marks and its inverse for the symmetric group of degree 5 are recalled. The table of USCIs of this group is obtained.

1. Introduction

The Pólya–Redfield theorem [1,2] and the Read–Redfield superposition theorem [2,3] have long been standard methods for enumerating graphs and other objects. Further extended formulations and accumulated results of enumerations have been described in Harary's textbook [4] and in several monographs [5,6]. Another methodology based on the concept of double coset has been applied to chemical enumeration [7,8].

Calculating the number of graphs (or other objects) with a given automorphism group has been reported to require tables of marks [9–11], which were once described in Burnside's textbook [12]. An alternative method based on the tables of marks has been developed to solve this type of problem [13]. Different approaches by using double cosets and framework groups [14] and by combining double cosets and tables of marks [15] have been applied to chemical enumeration.

We have reported promising methods based on the concept of *unit subduced cycle indices* (USCIs), which are calculated by subduction of coset representations [16,17]. By starting from the USCIs, we have successively derived subduced cycle indices (SCIs) [16], partial cycle indices (PCIs) [18], and a cycle index (CI) [19], all of which are useful in obtaining various generating functions. The SCIs and the

*Dedicated to Professor Frank Harary.

PCIs have been applied to enumeration of objects with given symmetries [20]. The CI has been proved to be equivalent to Pólya's cycle index, although they are different in their explicit forms [19].

Alternatively, we have presented the *elementary superposition* theorem concerning the SCIs [21,17,18], which provides a general method without relying on generating functions. On the basis of the elementary superposition theorem, we have derived the *partial superposition* theorem of the PCIs for calculating the number of objectes with a given symmetry as welll as the *superposition* of the CI for the total number of such objects [17]. The superposition of the CI has been proved to be equivalent to the Read–Redfield superposition theorem [18].

Although the USCI approach is applicable to various enumerations, previous examples have been mainly selected from chemical fields. In order to clarify the potentiality of the methods of the USCI approach, the present paper deals with enumeration of digraphs with a given automorphism group. In particular, (1) we precalculate a table of USCIs for the symmetric group of degree 5; thereby, (2) we enumerate five-vertex digraphs that are characterized by respective automorphism groups. In addition, we apply the elementary and partial superpositions to the same enumeration.

2. Formulation

Although the previous propositions of the USCI approach have mainly taken account of point groups [16], they are applicable to any groups of finite order without any modification. For enumerating digraphs, we will restate the propositions on the basis of symmetric groups and related ones in this paper.

Let $S^{[n]}$ be the symmetric group acting on $\Delta = \{1, 2, \ldots, n\}$. When we select one representative from every set of conjugated subgroups, we have a finite number of such representatives, which are denoted by $S_i^{[n]}$ ($i = 1, 2, \ldots, s$). Thereby, we have s coset representations (CRs) represented by $S^{[n]}(/S_i^{[n]})$.

Let Δ' be a set of $|\Delta'|$ objects. Suppose that the $S^{[n]}$ group acts on Δ' by acting on Δ to produce a permutation representation P on Δ', where n is equal to $|\Delta|$. According to Burnside [12], the P representation can be reduced into a sum of the CRs, as represented by

$$\mathrm{P} = \sum_{i=1}^{s} \alpha_i S^{[n]}(/S_i^{[n]}), \tag{1}$$

where the symbol α_i denotes the multiplicity of the CR. Each of the CRs, $S^{[n]}(/S_i^{[n]})$, corresponds to an orbit produced by the action of $S^{[n]}$ on Δ'. The multiplicities α_i are determined by using a table of marks, which is described in Burnside's textbook [12]. From eq. (1), we have the following equation concerning the lengths of orbits:

$$\sum_{i=1}^{s} \frac{\alpha_i |S^{[n]}|}{|S_i^{[n]}|} = |\Delta'|. \tag{2}$$

Let $Z(S^{[n]}(/S_i^{[n]}) \downarrow S_j^{[n]}; s_{d_{jk}})$ be a unit subduced cycle index (USCI) for the subduced $S^{[n]}(/S_i^{[n]}) \downarrow S_j^{[n]}$ (definition 1 of ref. [16]). Each USCI is a monomial determined for i and j; it can be precalculated by the data concerning the structure of $S^{[n]}$. On the basis of such USCIs (for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, s$), we define a subduced cycle index (SCI) [16], a partial cycle index (PCI) [18], and a cycle index (CI) [19].

Here, we regard the set Δ' as a domain which corresponds to a set of edges (or vertices) to be examined. It should be noted that the set Δ' is partitioned into several orbits in agreement with eq. (1). Let us consider a codomain:

$$X = \{X_1, X_2, \ldots X_{|X|}\}.$$

Consider a function from Δ' to X, where X_l takes a weight $w(X_l)$, which is selected to be an appropriate value according to a problem to be solved [16]. Suppose that θ_l of X_l's $(l = 1, 2, \ldots, |X|)$ are selected from the codomain X, where θ_l's satisfy a partition represented by

$$[\theta] : \theta_1 + \theta_2 + \ldots + \theta_{|X|} = |\Delta'|. \tag{3}$$

The function $f : \Delta' \to X$ has a weight represented by

$$W_\theta = w(X_1)^{\theta_1} w(X_2)^{\theta_2} \ldots w(X_{|X|})^{\theta_{|X|}}.$$

When all of such functions are collected to give a set represented by $F^{[\theta]} = \{f_1^{[\theta]}, f_2^{[\theta]}, \ldots, f_r^{[\theta]}\}$, the symmetric group $S^{[n]}$ acts on $F^{[\theta]}$ through the simultaneous action on Δ and Δ'.

Let the symbol A_{θ_i} be the number of non-equivalent functions (e.g. digraphs, etc.) with W_θ and $S_i^{[n]}$. This enumeration is to count orbits contained in $F^{[\theta]}$ during such an action. The A_{θ_i} value is evaluated by the following theorem.

THEOREM 1 (Theorem 4 of ref. [16]). (The number of non-equivalent functions)

$$A_{\theta i} = \sum_{j=1}^{s} \rho_{\theta j} \overline{m}_{ji} \quad (i = 1, 2, \ldots, s), \tag{4}$$

where $\overline{m}_{ji}$ is the ji-element in the inverse matrix of a mark table for $S^{[n]}$.

The number $(\rho_{\theta j})$ of fixed functions having W_θ and $S_j^{[n]}$ is evaluated by the following lemma.

LEMMA 1 (Lemma 1 of ref. [16]). (Evaluation of $\rho_{\theta j}$)

A set of such numbers ($\rho_{\theta j}$'s) for the $S_j^{[n]}$ automorphism group is given as the coefficients of a generating function,

$$\sum_{[\theta]} \rho_{\theta j} W_\theta = \mathrm{ZI}(\mathrm{S}_j^{[n]};\ s_{d_{jk}}) = \prod_{i=1}^{s} \prod_{\substack{\alpha=1 \\ (\alpha_i \neq 0)}}^{\alpha_i} Z(\mathrm{S}^{[n]}(/S_i^{[n]}) \downarrow S_j^{[n]};\ s_{d_{jk}}) \tag{5}$$

for $j = 1, 2, \ldots, s$, where the SCI of the right-hand side is replaced by a figure-inventory

$$s_{d_{jk}} = \sum_{l=1}^{|\mathrm{X}|} w(X_l)^{d_{jk}}. \tag{6}$$

By combining theorem 1 and lemma 1, we can easily obtain the following theorem, where we use the PCI defined above.

THEOREM 2 (Theorem 16.3 of ref. [17])

A generating function for calculating $A_{\theta i}$ is represented by

$$\sum_{[\theta]} A_{\theta i} W_\theta = \mathrm{PCI}(\mathrm{S}_i^{[n]};\ s_{d_{jk}}) = \sum_{j=1}^{s} \overline{m}_{ji} \mathrm{ZI}(\mathrm{S}_j^{[n]};\ s_{d_{jk}}) \tag{7}$$

for $i = 1, 2, \ldots, s$, where $s_{d_{jk}}$ is replaced by the figure-inventory represented by eq. (6).

This is a generating-function version of lemma 1 and theorem 1 [16, 17]. Kerber and Thürling [22] have alternatively derived a similar equation, though their formulation lacks the concepts of USCI, SCI and PCI.

Finally, we obtain the following theorem.

THEOREM 3 (Theorem 4 of ref. [19])

Let A_θ be the number of non-equivalent functions with the weight W_θ. A generating function of A_θ is represented by

$$\sum_{[\theta]} A_\theta W_\theta = \mathrm{CI}(\mathrm{S}^{[n]};\ s_{d_{jk}}) = \sum_{j=1}^{s} \left(\sum_{i=1}^{s} \overline{m}_{ji} \right) \mathrm{ZI}(\mathrm{S}_j^{[n]};\ s_{d_{jk}}), \tag{8}$$

where $s_{d_{jk}}$ is replaced by the figure-inventory represented by eq. (6).

3. The table of USCIs for the symmetric group of degree 5

The USCI approach requires precalculated tables of USCIs and relevant ones. We have already reported tables of USCIs for various point groups; C_2, C_3 (isomorphic to the alternating group of degree 3: $A^{[3]}$), C_s, C_i, S_4, C_{2v}, C_{3v} (isomorphic to the symmetric group of degree 3: $S^{[3]}$), C_{2h}, C_{3h}, D_2, D_3, D_{2h}, D_{3h}, D_{2d}, T (isomorphic to the alternating group of degree 4: $A^{[4]}$), T_d (isomorphic to the symmetric group of degree 4: $S^{[4]}$), and I_h.* Since isomorphic groups afford the same tables of marks, the same inverses, and the same tables of USCIs, such tables as reported for point groups [17] can also be employed to solve problems concerning permutation groups.

Sheehan [10] has reported the table of marks for the symmetric group of degree 5 ($S^{[5]}$), from which its inverse matrix (table 1) is easily obtained. For the purpose of enumerating five-vertex digraphs, we have to precalculate the table of USCI of this group. Let $S^{[5]}$ permute the set of integers {1, 2, 3, 4, 5}. Then we have distinct, up to conjugacy, subgroups of $S^{[5]}$ as follows:

$S_1^{[5]} = C_1$:	the identity permutation group;	order 1
$S_2^{[5]} = C_2$:	{(1)(2)(3)(4)(5), (14)(23)(5)};	order 2
$S_3^{[5]} = C_s$:	$S^{[2]} \times S^{[1]} \times S^{[1]} \times S^{[1]}$ = {(1)(2)(3)(4)(5), (1)(23)(4)(5)};	order 2
$S_4^{[5]} = C_3$:	$A^{[3]} \times S^{[1]} \times S^{[1]}$ = {(1)(2)(3)(4)(5), (132)(4)(5),(123)(4)(5)};	order 3
$S_5^{[5]} = S_4$:	⟨(1243)(5)⟩;	order 4
$S_6^{[5]} = C_{2v}$:	⟨(14)(23)(5), (1)(23)(4)(5)⟩;	order 4
$S_7^{[5]} = D_2$:	{(1)(2)(3)(4)(5), (13)(24)(5), (12)(34)(5), (14)(23)(5)};	order 4
$S_8^{[5]} = C_5$:	⟨(15432)⟩;	order 5
$S_9^{[5]} = C_{3v}$:	$S^{[3]} \times S^{[1]} \times S^{[1]}$;	order 6
$S_{10}^{[5]} = C_{3h}$:	$A^3 \times S^{[2]}$;	order 6
$S_{11}^{[5]} = D_3$:	{(1)(2)(3)(4)(5), (132)(4)(5), (123)(4)(5), (12)(3)(45), (1)(23)(45), (13)(2)(45)};	order 6
$S_{12}^{[5]} = D_{2d}$:	{(1)(2)(3)(4)(5), (13)(24)(5), (12)(34)(5), (14)(23)(5)(1423)(5), (1)(2)(34)(5), (12)(3)(4)(5), (1324)(5)};	order 8
$S_{13}^{[5]} = D_5$:	a dihedral group of order 10;	order 10
$S_{14}^{[5]} = T$:	$A^{[4]} \times S^{[1]}$;	order 12
$S_{15}^{[5]} = D_{3h}$:	$S^{[3]} \times S^{[2]}$;	order 12
$S_{16}^{[5]} = D_{5m}$:	⟨(1243)(5), (15432)⟩;	order 20
$S_{17}^{[5]} = T_d$:	$S^{[4]} \times S^{[1]}$;	order 24
$S_{18}^{[5]} = I$:	$A^{[5]}$;	order 60
$S_{19}^{[5]} =$	$S^{[5]}$;	order 120

*For a brief collection, see appendices A to E of ref. [17]. We developed a computer program for calculating mark tables, their inverses, and tables of USCIs, which was installed in a VAX11-750 computer (Digital Equipment). We have already obtained the tables for most point groups in addition to the ones listed here. These results will be published elsewhere.

Table 1

Inverse of the mark table for the $S^{[5]}$ group[a]

	$S^{[5]}$ $(/C_1)$	$S^{[5]}$ $(/C_2)$	$S^{[5]}$ $(/C_s)$	$S^{[5]}$ $(/C_3)$	$S^{[5]}$ $(/S_4)$	$S^{[5]}$ $(/C_{2v})$	$S^{[5]}$ $(/D_2)$	$S^{[5]}$ $(/C_5)$	$S^{[5]}$ $(/C_{3v})$	$S^{[5]}$ $(/C_{3h})$	$S^{[5]}$ $(/D_3)$	$S^{[5]}$ $(/D_{2d})$	$S^{[5]}$ $(/D_5)$	$S^{[5]}$ $(/T)$	$S^{[5]}$ $(/D_{3h})$	$S^{[5]}$ $(/D_{5m})$	$S^{[5]}$ $(/T_d)$	$S^{[5]}$ $(/I)$	$S^{[5]}$ $(/S^{[5]})$	Sum[b]
	$\times \frac{1}{120}$ for all elements																			
C_1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
C_2	−15	30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	15
C_s	−10	0	20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	10
C_3	−10	0	0	30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	20
S_4	0	−30	0	0	60	0	0	0	0	0	0	0	0	0	0	0	0	0	0	30
C_{2v}	30	−30	−60	0	0	60	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D_2	10	−30	0	0	0	0	20	0	0	0	0	0	0	0	0	0	0	0	0	0
C_5	−6	0	0	0	0	0	0	30	0	0	0	0	0	0	0	0	0	0	0	24
C_{3v}	30	0	−60	−30	0	0	0	0	60	0	0	0	0	0	0	0	0	0	0	0
C_{3h}	10	0	−20	−30	0	0	0	0	0	60	0	0	0	0	0	0	0	0	0	20
D_3	30	−60	0	−30	0	0	0	0	0	0	60	0	0	0	0	0	0	0	0	0
D_{2d}	0	60	0	0	−60	−60	−60	0	0	0	0	120	0	0	0	0	0	0	0	0
D_5	30	−60	0	0	0	0	0	−30	0	0	0	0	60	0	0	0	0	0	0	0
T	20	0	0	−60	0	0	−20	0	0	0	0	0	0	60	0	0	0	0	0	0
D_{3h}	−60	60	120	60	0	−120	0	0	−60	−60	−60	0	0	0	120	0	0	0	0	0
D_{5m}	0	60	0	0	−120	0	0	0	0	0	0	0	−60	0	0	120	0	0	0	0
T_d	−60	0	120	60	0	0	60	0	−120	0	0	−120	0	−60	0	0	120	0	0	0
I	−60	120	0	60	0	0	0	0	0	0	−60	0	−60	−60	0	0	0	60	0	0
$S^{[5]}$	60	−120	−120	−60	120	120	0	0	120	0	60	0	60	60	−120	−120	−120	−60	120	0

[a]Each element should be multiplied by 1/120.

[b]Sum = 120 = $\sum_{i=1}^{s} \overline{m}_{ji}$.

These subgroups, except $S^{[5]}_{16}$ and $S^{[5]}$ itself, are isomorphic to the point groups listed. The group $S^{[5]}_{16}$ (=D_{5m}) is a metacyclic group of order 20 generated by (1243)(5) and (15432).* For the sake of simplicity, we use point-group symbols (Schönflies' symbols) to denote the subgroups of $S^{[n]}$ as shown in the above list; for example, we employ $S^{[5]}_{18} = I$ in place of $S^{[5]}_{18} \cong I$. Although this convention may lack mathematical strictness, it is useful for discussing graph enumeration as a continuation of compound enumeration described in previous papers [17].

For the subduction of each coset representation into each subgroup, we construct a *subduced mark table* (SMT) [17, ch. 9] by selecting marks from the table of marks for the $S^{[5]}$ group. For example, let us calculate $S^{[5]}(/S^{[5]}_i) \downarrow D_{5m}$. Since the subgroup D_{5m} contains C_1, C_2, S_4, C_5, D_5, and D_{5m} as its subgroups, we collect the corresponding columns in the mark table of the $S^{[5]}$ group to obtain an SMT, which is multiplied by the inverse ($M^{-1}_{D_{5m}}$) of a mark table for the D_{5m} group.

$$M^{-1}_{D_{5m}} = \begin{pmatrix} 1/20 & 0 & 0 & 0 & 0 & 0 \\ -1/4 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & 1 & 0 & 0 & 0 \\ -1/20 & 0 & 0 & 1/4 & 0 & 0 \\ 1/4 & -1/2 & 0 & -1/4 & 1/2 & 0 \\ 0 & 1/2 & -1 & 0 & -1/2 & 1 \end{pmatrix}. \tag{9}$$

Thus, we have

$$\begin{pmatrix} 120 & 0 & 0 & 0 & 0 & 0 \\ 60 & 4 & 0 & 0 & 0 & 0 \\ 60 & 0 & 0 & 0 & 0 & 0 \\ 40 & 0 & 0 & 0 & 0 & 0 \\ 30 & 2 & 2 & 0 & 0 & 0 \\ 30 & 2 & 0 & 0 & 0 & 0 \\ 30 & 6 & 0 & 0 & 0 & 0 \\ 24 & 0 & 0 & 4 & 0 & 0 \\ 20 & 0 & 0 & 0 & 0 & 0 \\ 20 & 0 & 0 & 0 & 0 & 0 \\ 20 & 4 & 0 & 0 & 0 & 0 \\ 15 & 3 & 1 & 0 & 0 & 0 \\ 12 & 4 & 0 & 2 & 2 & 0 \\ 10 & 2 & 0 & 0 & 0 & 0 \\ 10 & 2 & 0 & 0 & 0 & 0 \\ 6 & 2 & 2 & 1 & 1 & 1 \\ 5 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \times M^{-1}_{D_{5m}} = \begin{pmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{10}$$

*The list of subgroups of $S^{[5]}$ is identical with the reported in ref. [10] except notation. The symbol D_{5m} is coined after its behavior, where the subscript (m) stems from "minus", because the group contains all the elements of D_5 as well as the same number of elements of minus parity.

Table 2

Table of USCIs for the $S^{[5]}$ group.

	$\downarrow C_1$	$\downarrow C_2$	$\downarrow C_s$	$\downarrow C_3$	$\downarrow S_4$	$\downarrow C_{2v}$	$\downarrow D_2$	$\downarrow C_5$	$\downarrow C_{3v}$	$\downarrow C_{3h}$	$\downarrow D_3$	$\downarrow D_{2d}$	$\downarrow D_5$	$\downarrow T$	$\downarrow D_{3h}$	$\downarrow D_{5m}$	$\downarrow T_d$	$\downarrow I$	$\downarrow S^{[5]}$
$S^{[5]}(/C_1)$	s_1^{120}	s_2^{60}	s_2^{60}	s_3^{40}	s_4^{30}	s_4^{30}	s_4^{30}	s_5^{24}	s_6^{20}	s_6^{20}	s_6^{20}	s_8^{15}	s_{10}^{12}	s_{12}^{10}	s_{12}^{10}	s_{20}^{6}	s_{24}^{5}	s_{60}^{2}	s_{120}
$S^{[5]}(/C_2)$	s_1^{60}	$s_1^4 s_2^{28}$	s_2^{30}	s_3^{20}	$s_2^2 s_4^{14}$	$s_2^2 s_4^{14}$	$s_2^6 s_4^{12}$	s_5^{12}	s_6^{10}	s_6^{10}	$s_3^4 s_6^8$	$s_4^3 s_8^6$	$s_5^4 s_{10}^4$	$s_6^2 s_{12}^4$	$s_6^2 s_{12}^4$	$s_{10}^2 s_{20}^2$	$s_{12} s_{24}^2$	s_{30}^2	s_{60}
$S^{[5]}(/C_s)$	s_1^{60}	s_2^{30}	$s_1^6 s_2^{27}$	s_3^{20}	s_4^{15}	$s_2^6 s_4^{12}$	s_4^{15}	s_5^{12}	$s_3^6 s_6^7$	$s_3^2 s_6^9$	s_6^{10}	$s_4^3 s_8^6$	s_{10}^6	s_{12}^5	$s_6^4 s_{12}^3$	s_{20}^3	$s_{12}^3 s_{24}$	s_{60}	s_{60}
$S^{[5]}(/C_3)$	s_1^{40}	s_2^{20}	s_2^{20}	$s_1^4 s_3^{12}$	s_4^{10}	s_4^{10}	s_4^{10}	s_5^8	$s_2^2 s_6^6$	$s_2^2 s_6^6$	$s_2^2 s_6^6$	s_8^5	s_{10}^4	$s_4^4 s_{12}^2$	$s_4 s_{12}^3$	s_{20}^2	$s_8^2 s_{24}$	s_{20}^2	s_{40}
$S^{[5]}(/S_4)$	s_1^{30}	$s_1^2 s_2^{14}$	s_2^{15}	s_3^{10}	$s_1^2 s_4^7$	$s_2 s_4^7$	$s_2^3 s_4^6$	s_5^6	s_6^5	s_6^5	$s_3^2 s_6^4$	$s_2 s_4 s_8^3$	$s_5^2 s_{10}^2$	$s_6 s_{12}^2$	$s_6 s_{12}^2$	$s_{10}^2 s_{20}$	$s_6 s_{24}$	s_{30}	s_{30}
$S^{[5]}(/C_{2v})$	s_1^{30}	$s_1^2 s_2^{14}$	$s_1^6 s_2^{12}$	s_3^{10}	$s_2 s_4^7$	$s_1^2 s_2^4 s_4^5$	$s_2^3 s_4^6$	s_5^6	$s_3^6 s_6^2$	$s_3^2 s_6^4$	$s_3^2 s_6^4$	$s_2 s_4^3 s_8^2$	$s_5^2 s_{10}^2$	$s_6 s_{12}^2$	$s_3^2 s_6^2 s_{12}$	$s_{10}^2 s_{20}$	$s_6 s_{12}^2$	s_{30}	s_{30}
$S^{[5]}(/D_2)$	s_1^{30}	$s_1^5 s_2^{12}$	s_2^{15}	s_3^{10}	$s_2^3 s_4^6$	$s_2^3 s_4^6$	$s_1^6 s_4^6$	s_5^6	s_6^5	s_6^5	$s_3^6 s_6^2$	$s_2^3 s_8^3$	s_5^6	$s_3^2 s_{12}^2$	$s_6^3 s_{12}$	s_{10}^3	$s_6 s_{24}$	s_{15}^2	s_{30}
$S^{[5]}(/C_5)$	s_1^{24}	s_2^{12}	s_2^{12}	s_3^8	s_4^6	s_4^6	s_4^6	$s_1^4 s_5^4$	s_6^4	s_6^4	s_6^4	s_8^3	$s_2^2 s_{10}^2$	s_{12}^2	s_{12}^2	$s_4 s_{20}$	s_{24}	s_{12}^2	s_{24}
$S^{[5]}(/C_{3v})$	s_1^{20}	s_2^{10}	$s_1^6 s_2^7$	$s_1^2 s_3^6$	s_4^5	$s_2^6 s_4^2$	s_4^5	s_5^4	$s_1^2 s_3^4 s_6$	$s_2 s_3^2 s_6^2$	$s_2 s_6^3$	$s_4^3 s_8$	s_{10}^2	$s_4^2 s_{12}$	$s_2 s_6^3$	s_{20}	$s_4^2 s_{12}$	s_{20}	s_{20}
$S^{[5]}(/C_{3h})$	s_1^{20}	s_2^{10}	$s_1^2 s_2^9$	$s_1^2 s_3^6$	s_4^5	$s_2^2 s_4^4$	s_4^5	s_5^4	$s_2 s_3^2 s_6^2$	$s_1^2 s_6^3$	$s_2 s_6^3$	$s_4 s_8^2$	s_{10}^2	$s_4^2 s_{12}$	$s_2 s_6 s_{12}$	s_{20}	$s_8 s_{12}$	s_{20}	s_{20}
$S^{[5]}(/D_3)$	s_1^{20}	$s_1^4 s_2^8$	s_2^{10}	$s_1^2 s_3^6$	$s_2^2 s_4^4$	$s_2^2 s_4^4$	$s_2^6 s_4^2$	s_5^4	$s_2 s_6^3$	$s_2 s_6^3$	$s_1^2 s_3^2 s_6^2$	$s_4^3 s_8$	s_5^4	$s_4^2 s_6^2$	$s_2 s_6 s_{12}$	s_{10}^2	$s_8 s_{12}$	s_{10}^2	s_{20}
$S^{[5]}(/D_{2d})$	s_1^{15}	$s_1^3 s_2^6$	$s_1^3 s_2^6$	s_3^5	$s_1 s_2 s_4^3$	$s_1 s_2^3 s_4^2$	$s_1^3 s_4^3$	s_5^3	$s_3^3 s_6$	$s_3 s_6^2$	$s_3^3 s_6$	$s_1 s_2 s_4 s_8$	s_5^3	s_3^{12}	$s_3 s_6^2$	$s_5 s_{10}$	$s_3 s_{12}$	s_{15}	s_{15}
$S^{[5]}(/D_5)$	s_1^{12}	$s_1^4 s_2^4$	s_2^6	s_3^4	$s_2^2 s_4^2$	$s_2^2 s_4^2$	s_2^6	$s_1^2 s_5^2$	s_6^2	s_6^2	s_3^4	s_4^3	$s_1^2 s_5^2$	s_6^2	s_6^2	$s_2 s_{10}$	s_{12}	s_6^2	s_{12}
$S^{[5]}(/T)$	s_1^{10}	$s_1^2 s_2^4$	s_2^5	$s_1^4 s_3^2$	$s_2 s_4^2$	$s_2 s_4^2$	$s_1^2 s_4^2$	s_5^2	$s_2^2 s_6$	$s_2^2 s_6$	$s_2^2 s_3^2$	$s_2 s_8$	s_5^2	$s_1^2 s_4^2$	$s_4 s_6$	s_{10}	$s_2 s^8$	s_5^2	s_{10}
$S^{[5]}(/D_{3h})$	s_1^{10}	$s_1^2 s_2^4$	$s_1^4 s_2^3$	$s_1 s_3^3$	$s_2 s_4^2$	$s_1^2 s_2^2 s_4$	$s_2^3 s_4$	s_5^2	$s_1 s_3^3$	$s_1 s_3 s_6$	$s_1 s_3 s_6$	$s_2 s_4^2$	s_5^2	$s_4 s_6$	$s_1 s_3 s_6$	s_{10}	$s_4 s_6$	s_{10}	s_{10}
$S^{[5]}(/D_{5m})$	s_1^6	$s_1^2 s_2^2$	s_2^3	s_3^2	$s_1^2 s_4$	$s_2 s_4$	s_2^3	$s_1 s_5$	s_6	s_6	s_3^2	$s_2 s_4$	$s_1 s_5$	s_6	s_6	$s_1 s_5$	s_6	s_6	s_6
$S^{[5]}(/T_d)$	s_1^5	$s_1 s_2^2$	$s_1^3 s_2$	$s_1^2 s_3$	$s_1 s_4$	$s_1 s_2^2$	$s_1 s_4$	s_5	$s_1^2 s_3$	$s_2 s_3$	$s_2 s_3$	$s_1 s_4$	s_5	$s_1 s_4$	$s_2 s_3$	s_5	$s_1 s_4$	s_5	s_5
$S^{[5]}(/I)$	s_1^2	s_1^2	s_2	s_1^2	s_2	s_2	s_1^2	s_1^2	s_2	s_2	s_1^2	s_2	s_1^2	s_1^2	s_2	s_2	s_2	s_1^2	s_2
$S^{[5]}(/S^{[5]})$	s_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1
sum[a]	$\frac{1}{120}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	0	0	$\frac{1}{5}$	0	$\frac{1}{6}$	0	0	0	0	0	0	0	0	0

[a] Sum $= \sum_{i=1}^{s} \overline{m}_{ji}$. See the rightmost column.

in which each row of the last matrix indicates multiplicities of the corresponding CR of D_{5m}. For example, the second row of the matrix corresponds to the subduction represented by

$$S^{[5]}(/C_2) \downarrow D_{5m} = 2D_{5m}(/C_1) + 2D_{5m}(/C_2). \tag{11}$$

In accord with this equation, the subduction is characterized by USCI $s_{10}^2 s_{20}^2$, in which the subscripts come from the relationships $|D_{5m}| / |C_1| = 20/1 = 20$ and $|D_{5m}| / |C_2| = 20/2 = 10$, and the powers stem from the coefficients of the CRs. With the data of this matrix, we have the $\downarrow D_{5m}$ column of table 2. The other columns of table 2 are calculated in the same line.

4. Enumeration of digraphs

Suppose $\Delta^{[2]}$ is the set that contains all of the ordered pairs $[p\ q]$, where p and q are selected from Δ so as to be different. The action of $S^{[n]}$ on Δ is accompanied by the action of $S^{[n]}$ on $\Delta^{[2]}$, which creates a permutation representation. This formulation is essentially equivalent to Harary's formulation for digraph enumeration [4], except that the present one considers such a permutation *representation* in place of a reduced ordered pair *group*.

Here, we regard the set $\Delta^{[2]}$ as a domain in place of Δ' described above. Let us consider a codomain $X = \{X_1, X_2\}$. Consider a function from $\Delta^{[2]}$ to X, which satisfies $f([p\ q]) = X_1$ if no edge is present between p and q, and $f([p\ q]) = X_2$ if there is an arrow directed from p to q. Enumeration of digraphs is formulated as enumeration of such non-equivalent functions. Suppose that θ_1 of X_1 and θ_2 of X_2 are selected from the codomain X, where we have $[\theta] : \theta_1 + \theta_2 = |\Delta^{[2]}|$. The function $f: \Delta^{[2]} \to X$ has a weight represented by $W_\theta = w(X_1)^{\theta_1} w(X_2)^{\theta_2}$. By means of this formulation, we are able to apply the above propositions to this case.

4.1. ASSIGNMENT OF ORBITS TO COSET REPRESENTATIONS

For enumerating five-vertex digraphs, we first find orbits of a domain and assign them to coset representations. Let us consider a set represented by $\Delta = \{1, 2, 3, 4, 5\}$, which is permuted by a permutation of $S^{[5]}$. As described above, we construct $\Delta^{[2]}$ on the basis of the Δ set, where the size of $\Delta^{[2]}$ ($|\Delta^{[2]}|$) is equal to 20. The $\Delta^{[2]}$ set is considered to be a set of directed edges, which is recognized as the present domain. We then count fixed points (marks) during the operations of every subgroup. For example, the permutation (1)(23)(4)(5) of C_s keeps six ordered pairs invariant, i.e. [1 4], [4 1], [1 5], [5 1], [4 5], and [5 4]; hence, the mark of P and C_s is calculated to be equal to 6 ($\mu_3 = 6$). This operation is repeated for every subgroup ($S_1^{[5]} - S_{19}^{[5]}$). The resulting μ_j values ($j = 1, 2, \ldots, 19$) are collected to form a row vector, which is called a fixed point vector (FPV):

$$\text{FPV} = (20, 0, 6, 2, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Hence, we have

$$(20, 0, 6, 2, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0)M^{-1}$$
$$= (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$

where M^{-1} is the inverse matrix listed in table 1 [17, ch. 5]. The resulting vector indicates the appearance of the $S^{[5]}(/C_{3v})$ CR $(= S^{[5]}(/S_9^{[5]}))$. Note that $|S^{[5]}|/|C_{3v}|$ is equal to 20, because $|S^{[5]}| = 120$ and $|C_{3v}| = 6$. The C_{3v} group $(= S^{[3]} \times S^{[1]} \times S^{[1]}$ in the present terminology) that appears in the parentheses in the notation of the CR can be proved to be the stabilizer of a special one of the ordered pairs in $\Delta^{[2]}$ [23].

In general, we have the following theorem.

THEOREM 4

Let $\Delta^{[2]}$ be the set of all of the ordered pairs $[p\ q]$ $(p \neq q)$ that are selected from $\Delta = \{1, 2, \ldots, n\}$. The set $\Delta^{[2]}$ has only one orbit that is goverened by a coset representation represented by $S^{[n]}(/H)$, where $H = S^{[n-2]} \times S^{[1]} \times S^{[1]}$. Obviously, the degree of the $S^{[n]}(/H)$ CR is equal to $|S^{[n]}|/|H| = n(n-1)$.

Proof

Consider an ordered pair $[p\ q]$ selected from $\Delta^{[2]}$. Let h be a permutation that keeps the pair invariant. Since the remaining elements of $\Delta^{[2]}$ are permuted by each permutation of $S^{[n-2]}$, the h is represented by $h' \times (p)(q)$, where $h' \in S^{[n-2]}$. Conversely, if h is represented by $h' \times (p)(q)$, where $h' \in S^{[n-2]}$, the h permutation keeps the $[p\ q]$ pair invariant. Hence, $H = S^{[n-2]} \times S^{[1]} \times S^{[1]}$ is the stabilizer of $[p\ q]$. Obviously, this is also the stabilzer of $[q\ p]$.

Similarly, we have $H' = S'^{[n-2]} \times S'^{[1]} \times S'^{[1]}$ as the stabilizer of another ordered pair $[p', q']$. Obviously, H is different from H′, and they can be proved to be conjugate to each other. This means that the set $\Delta^{[2]}$ has one orbit governed by $S^{[n]}(/H)$. Equation (2) affords $|S^{[n]}|/|H| = n(n-1)$ because $|S^{[n]}| = n!$ and $|H| = (n-2)!$ □

Suppose that $\Delta^{\{2\}}$ is the set that contains all of the unordered pairs $\{p\ q\}$, where p and q are selected from $\Delta = \{1, 2, \ldots, n\}$ so as to be different. The action of $S^{[n]}$ on Δ is accompanied by the action of $S^{[n]}$ on $\Delta^{\{2\}}$, which creates a permutation representation. Let us consider the case of $S^{[5]}$ acting on $\Delta = \{1, 2, 3, 4, 5\}$. Obviously, we have $|\Delta^{\{2\}}| = 10$ in this case. The corresponding FPV is calculated to be

$$\text{FPV} = (10, 2, 4, 1, 0, 2, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0).$$

The multiplication of the FPV by the inverse (table 1) afford a row vector

$$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0),$$

which indicates that the $\Delta^{\{2\}}$ has one orbit goverened by the CR $S^{[5]}(/D_{3h})$.

In general, we have the following theorem.

THEOREM 5

Let $\Delta^{(2)}$ be the set of all of the unordered pairs $\{p\ q\}$ $(p \neq q)$ that are selected from $\Delta = \{1, 2, \ldots, n\}$. The set $\Delta^{(2)}$ has only one orbit that is goverened by a coset representation represented by $S^{[n]}(/H)$, where $H = S^{[n-2]} \times S^{[2]}$. Obviously, the degree of the $S^{[n]}(/H)$ CR is equal to $|S^{[n]}| / |H| = n(n-1)/2$.

Proof

Consider an unordered pair $\{p\ q\}$ selected from $\Delta^{(2)}$. Let h be a permutation that keeps the pair invariant. When we focus our attention on the subset $\{p\ q\}$ of $\Delta^{(2)}$, it is kept invariant by the action $S^{[2]}$ $(= \{(p)(q), (pq)\})$. Since the remaining elements of $\Delta^{(2)}$ are permuted by each permutation of $S^{[n-2]}$, the h is represented by $h' \times h''$, where $h' \in S^{[n-2]}$ and $h'' \in S^{[2]}$. Conversely, if h is represented by $h' \times h''$, where $h' \in S^{[n-2]}$ and $h'' \in S^{[2]}$, the h permutation keeps the $\{p\ q\}$ pair. Hence, $H = S^{[n-2]} \times S^{[2]}$ is the stabilizer of $\{p\ q\}$.

Similarly, we have $H' = S'^{[n-2]} \times S'^{[2]}$ as the stabilizer of another ordered pair $\{p', q'\}$. Obviously, H is different from H′, and they can be proved to be conjugate to each other. This means that the set $\Delta^{(2)}$ has one orbit governed by $S^{[n]}(/H)$. The degree of the CR is calculated to be that eq. (2) affords $|S^{[n]}| / |H| = n(n-1)/2$, when $|S^{[n]}| = n!$ and $|H| = (n-2)! \times 2$ are introduced into eq. (2). □

This result can be used for the enumeration of graphs, which has been discussed by Sheehan [10]. The enumeration of graphs with a given automorphism group can be solved along the same lines as that of digraphs in the light of the present USCI approach.* Hence, we focus our attention on the enumeration of digraphs with a given automorphism group.

4.2. ENUMERATION BASED ON SCIS

The first method for enumerating five-vertex digraphs with an automorphism group is based on a generating function. Since the $\Delta^{[2]}$ domain is governed by the $S^{[5]}(/C_{3v})$ CR, we select the USCIs in the $S^{[5]}(/C_{3v})$ row of table 2. In the present case, the USCIs are equal to SCIs for enumerating ρ_j values.

$$C_1: \quad s_1^{20} = (1+x)^{20}, \tag{12}$$

$$C_2: \quad s_2^{10} = (1+x^2)^{10}, \tag{13}$$

$$C_s: \quad s_1^6 s_2^7 = (1+x)^6(1+x^2)^7, \tag{14}$$

*The enumeration of four-vertex graphs with a given automorphism group is equivalent to that of adamantane isomers (or homologs) with a given symmetry. See ref. [24].

$$C_3 : \quad s_1^2 s_3^6 = (1+x)^2(1+x^3)^6, \tag{15}$$

$$S_4 : \quad s_4^5 = (1+x^4)^5, \tag{16}$$

$$C_{2v} : \quad s_2^6 s_4^2 = (1+x^2)^6(1+x^4)^2, \tag{17}$$

$$D_2 : \quad s_4^5 = (1+x^4)^5, \tag{18}$$

$$C_5 : \quad s_5^4 = (1+x^5)^4, \tag{19}$$

$$C_{3v} : \quad s_1^2 s_3^4 s_6 = (1+x)^2(1+x^3)^4(1+x^6), \tag{20}$$

$$C_{3h} : \quad s_2 s_3^2 s_6^2 = (1+x^2)(1+x^3)^2(1+x^6)^2, \tag{21}$$

$$D_3 : \quad s_2 s_6^3 = (1+x^2)(1+x^6)^3, \tag{22}$$

$$D_{2d} : \quad s_4^3 s_8 = (1+x^4)^3(1+x^8), \tag{23}$$

$$D_5 : \quad s_{10}^2 = (1+x^{10})^2, \tag{24}$$

$$T : \quad s_4^2 s_{12} = (1+x^4)^2(1+x^{12}), \tag{25}$$

$$D_{3h} : \quad s_2 s_6^3 = (1+x^2)(1+x^6)^3, \tag{26}$$

$$D_{5m} : \quad s_{20} = 1+x^{20}, \tag{27}$$

$$T_d : \quad s_4^2 s_{12} = (1+x^4)^2(1+x^{12}), \tag{28}$$

$$I : \quad s_{20} = 1+x^{20}, \tag{29}$$

$$S^{[5]} : \quad s_{20} = 1+x^{20}. \tag{30}$$

The weights of X_1 and X_2 are expressed by $w(X_1) = 1$ for the absence of an edge and $w(X_2) = x$ for the presence of a directed edge. Hence, the figure-inventory (eq. (6)) is equal to $s_d = 1 + x^d$, which is introduced into each of the SCIs in the light of lemma 1. Expansion of the resulting generating function affords each of $\rho_{\theta j}$ values as the coefficient of the x^{θ_2} term (or more precisely of the $1^{\theta_1}x^{\theta_2}$ term) as shown in table 3, where $[\theta]$ is expressed by $\theta_1 + \theta_2 = 20$.

According to theorem 1, the data of table 3 are regarded as a matrix, which is multiplied by the inverse M^{-1} (table 1). The result is shown in table 4.

When we substitute 1 for x in eqs. (12)–(30) and collect the resulting values as a row vector, we obtain an FPV for calculating the total number of digraphs with a given automorphism group:

$$\text{FPV} = (2^{20}, 2^{10}, 2^{13}, 2^8, 2^5, 2^8, 2^5, 2^4, 2^7, 2^5, 2^4, 2^4, 2^2, 2^3, 2^4, 2, 2^3, 2, 2).$$

Table 3

Coefficients($\rho_{\theta j}$).

Term	C_1	C_2	C_s	C_3	S_4	C_{2v}	D_2	C_5	C_{3v}	C_{3h}	D_3	D_{2d}	D_5	T	D_{3h}	D_{5m}	T_d	I	$S^{[5]}$
x^0, x^{20}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
x^1, x^{19}	20	0	6	2	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0
x^2, x^{18}	190	10	22	1	0	6	0	0	1	1	1	0	0	0	1	0	0	0	0
x^3, x^{17}	1140	0	62	6	0	0	0	0	4	2	0	0	0	0	0	0	0	0	0
x^4, x^{16}	4845	45	141	12	5	17	5	0	8	0	0	3	0	2	0	0	2	0	0
x^5, x^{15}	15504	0	272	6	0	0	0	4	4	2	0	0	0	0	0	0	0	0	0
x^6, x^{14}	38760	120	456	15	0	32	0	0	7	3	3	0	0	0	3	0	0	0	0
x^7, x^{13}	77520	0	672	30	0	0	0	0	14	0	0	0	0	0	0	0	0	0	0
x^8, x^{12}	125970	210	882	15	10	46	10	0	7	3	3	4	0	1	3	0	1	0	0
x^9, x^{11}	167960	0	1036	20	0	0	0	0	8	4	0	0	0	0	0	0	0	0	0
x^{10}	184756	252	1092	40	0	52	0	6	16	0	0	0	2	0	0	0	0	0	0

Table 4

The numbers of five-vertex digraphs with a given automorphism group.

Term	C_1	C_2	C_s	C_3	S_4	C_{2v}	D_2	C_5	C_{3v}	C_{3h}	D_3	D_{2d}	D_5	T	D_{3h}	D_{5m}	T_d	I	$S^{[5]}$	Total
x^0, x^{20}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
x^1, x^{19}	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1
x^2, x^{18}	0	1	1	0	0	2	0	0	0	0	0	0	0	0	1	0	0	0	0	5
x^3, x^{17}	5	0	8	0	0	0	0	0	2	1	0	0	0	0	0	0	0	0	0	16
x^4, x^{16}	28	6	13	1	1	7	0	0	2	0	0	1	0	0	0	0	2	0	0	61
x^5, x^{15}	107	0	43	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	0	154
x^6, x^{14}	278	22	59	2	0	13	0	0	2	0	0	0	0	0	3	0	0	0	0	379
x^7, x^{13}	591	0	105	4	0	0	0	0	7	0	0	0	0	0	0	0	0	0	0	707
x^8, x^{12}	962	38	124	2	3	18	0	0	1	0	0	3	0	0	3	0	1	0	0	1155
x^9, x^{11}	1314	0	168	2	0	0	0	0	4	2	0	0	0	0	0	0	0	0	0	1490
x^{10}	1431	49	148	6	0	26	0	1	8	0	0	0	1	0	0	0	0	0	0	1670
total	8001	183	1190	28	8	106	0	3	50	8	0	8	1	0	14	0	6	0	2	9608

Note that the power of each element represents the number of suborbits that are generated by the corresponding subduction. In the light of corollary 3.2 of ref. [16], the FPV is multiplied by the inverse M^{-1} (table 1) to afford

$$(8001, 183, 1190, 28, 8, 106, 0, 3, 50, 8, 0, 8, 1, 0, 14, 0, 6, 0, 2).$$

These values are identical to the sums of the corresponding columns, as listed at the bottom of table 4. The total number is obtained to be 9608, which is vertified by the value shown in appendix II of ref. [4].

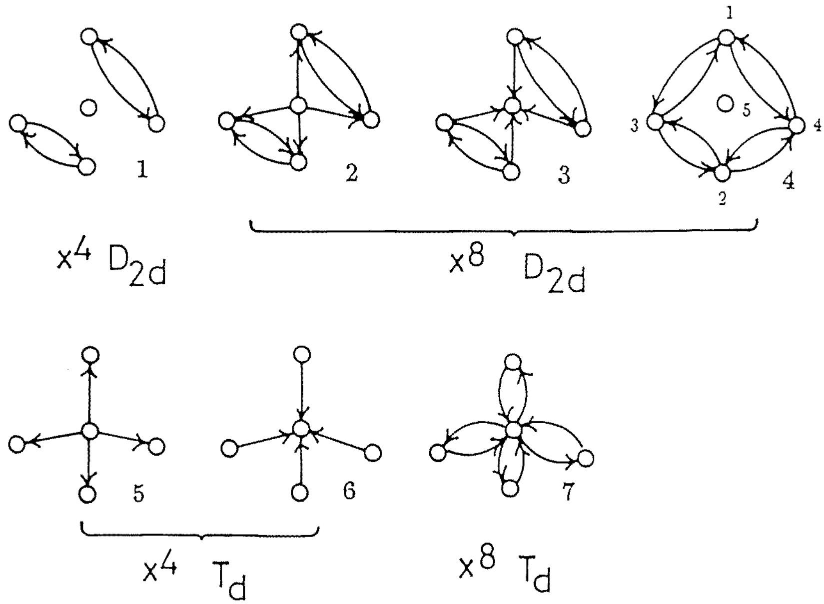

Fig. 1. Five-vertex digraphs with D_{2d} and T_d automorphism groups (x^{θ_2}, $\theta_2 \leq 10$).

For exemplifying the results of table 4, fig. 1 depicts lower digraphs (x^{θ_2}, $\theta_2 \leq 10$) with D_{2d} and T_d automorphism groups. These digraphs are drawn on the basis of a hypothetical methane skeleton (T_d point group) and its D_{2d} distorted form, in which four bonds and six edges are taken into consideration. Since this paper emphasizes correspondence between permutation groups and point groups, the automorphism groups of digraphs can be deduced by the inspection of such hypothetical skeletons.

It is worth mentioning why no digraphs emerge for several automorphism groups; inherent lack and accidental lack. Problems of this type have been discussed by Hässelbarth [13] and by Brocas [14]. Here, we treat such problems through an alternative approach in which the inherent lack is explained by comparing the SCI of an automorphism group at issue with the SCI of each of its subgroups. The lack of digraphs of I, D_{5m}, T, and D_3 is inherent. The SCI (s_{20}) of the I group is the same as that of the supergroup $S^{[5]}$; hence, attempted constitution of a digraph of the I automorphism group results in the inevitable appearance of an $S^{[5]}$ digraph. The same situation is true for D_{5m} and $S^{[5]}$ (s_{20}), for T and T_d ($s_4^2 s_{12}$), and for D_3 and D_{3h} ($s_2 s_6^3$).

On the other hand, the absence of digraphs of the D_2 automorphism group is accidental, because the SCI of D_2 (s_4^5) is different from any SCIs of its supergroups, i.e. $s_4^3 s_8$ of D_{2d} and $s_4^2 s_{12}$ of T (or T_d).

For illustrating such accidental lack, let us consider further coloring of a digraph 4 depicted in fig. 1. If we color the set of directed edges ([1 4], [4 1], [2 3] and [3 2]) red and the other set of directed edges ([1 3], [3 1], [2 4] and [4 2] green, we have a colored digraph of the D_2 automorphism group. This desymmetrication process reflects the difference between the SCI of the D_{2d} ($s_4^3 s_8$) and that of the D_2 (s_4^5). On the other hand, in the case of the inherent lack, there appears no digraph even if we consider such coloring. For example, the T_d digraphs (**5**, **6** and **7**) of fig. 1 by no means produce colored T digraphs.

Systematic examination of inherent lack and of accidental lack can be accomplished by desymmetrization lattices, which have been described for doing the same task concerning point groups [17].

Table 5 lists the numbers of four-vertex digraphs with a given automorphism group, which are calculated along the same line as above. Each of the subgroups belonging to $S^{[4]}$ is isomorphic to that of $S^{[5]}$ through the point-group symbol. The table of marks, its inverse and the table of USCIs for the T_d point group [17] are employed for the present $S^{[4]}$ case.

Table 5

The numbers of four-vertex digraphs with a given automorphism group.[a]

Term	C_1	C_2	C_s	C_3	S_4	D_2	C_{2v}	C_{3v}	D_{2d}	T	$S^{[4]}$	Total
x^0, x^{12}	0	0	0	0	0	0	0	0	0	0	1	1
x^1, x^{11}	0	0	1	0	0	0	0	0	0	0	0	1
x^2, x^{10}	1	1	2	0	0	0	1	0	0	0	0	5
x^3, x^9	7	0	3	1	0	0	0	2	0	0	0	13
x^4, x^8	16	2	6	0	1	0	1	0	1	0	0	27
x^5, x^7	28	0	10	0	0	0	0	0	0	0	0	38
x^6	32	4	6	2	0	0	2	2	0	0	0	48
total	136	10	50	4	2	0	6	6	2	0	2	218

[a]The $S^{[4]}$ group is isomorphic to the T_d point group.

All of the digraphs enumerated here have been depicted in appendix II of ref. [4], although they are not itemized with respect to automorphism groups. The assignment of an automorphism group to each of the digraphs can conveniently be accomplished by means of such a point-group symbol as listed in table 5. Since the $S^{[4]}$ group is isomorphic to the T_d point group, we employ a tetrahedron (T_d) as a parent, the six edges of which are substituted with arrows. This assignment will show the validity of table 5.

4.3. ENUMERATION BASED ON PCIS

An alternative method for calculating the number of digraphs with a given automorphism group is based on PCIs described in theorem 2. By using the inverse (table 1) and the USCIs (the $S^{[5]}(/C_{3v})$ row of table 2), we have PCIs for enumerating digraphs:

$$\begin{aligned}\mathrm{PCI}(\mathrm{C}_1; s_{d_{jk}}) = {} & \tfrac{1}{120}(s_1^{20} - 15s_2^{10} - 10s_1^6s_2^7 - 10s_1^2s_3^6 + 30s_2^6s_4^2 + 10s_4^5 - 6s_5^4 + 30s_1^2s_3^4s_6 \\ & + 10s_2s_3^2s_6^2 + 30s_2s_6^3 + 30s_{10}^2 + 20s_4^2s_{12} - 60s_2s_6^3 - 60s_4^2s_{12}),\end{aligned} \quad (31)$$

$$\mathrm{PCI}(\mathrm{C}_2; s_{d_{jk}}) = \tfrac{1}{4}(s_2^{10} - s_4^5 - s_2^6s_4^2 - s_4^5 - 2s_2s_6^3 + 2s_4^3s_8 - 2s_{10}^2 + 2s_2s_6^3 + 2s_{20}), \quad (32)$$

$$\mathrm{PCI}(\mathrm{C_s}; s_{d_{jk}}) = \tfrac{1}{6}(s_1^6s_2^7 - 3s_2^6s_4^2 - 3s_1^2s_3^4s_6 - s_2s_3^2s_6^2 + 6s_2s_6^3 + 6s_4^2s_{12} - 6s_{20}), \quad (33)$$

$$\mathrm{PCI}(\mathrm{C}_3; s_{d_{jk}}) = \tfrac{1}{4}(s_1^2s_3^6 - s_1^2s_3^4s_6 - s_2s_3^2s_6^2 - s_2s_6^3 - 2s_4^2s_{12} + 2s_2s_6^3 + 2s_4^2s_{12}), \quad (34)$$

$$\mathrm{PCI}(\mathrm{S}_4; s_{d_{jk}}) = \tfrac{1}{2}(s_4^5 - s_4^3s_8 - 2s_{20} + 2s_{20}), \quad (35)$$

$$\mathrm{PCI}(\mathrm{C_{2v}}; s_{d_{jk}}) = \tfrac{1}{2}(s_2^6s_4^2 - s_4^3 - 2s_2s_6^3 + 2s_{20}), \quad (36)$$

$$\mathrm{PCI}(\mathrm{D}_2; s_{d_{jk}}) = \tfrac{1}{6}(s_4^5 - 3s_4^3s_8 - s_4^2s_{12} + 3s_4^2s_{12}), \quad (37)$$

$$\mathrm{PCI}(\mathrm{C}_5; s_{d_{jk}}) = \tfrac{1}{4}(s_5^4 - s_{10}^2), \quad (38)$$

$$\mathrm{PCI}(\mathrm{C_{3v}}; s_{d_{jk}}) = \tfrac{1}{2}(s_1^2s_3^4s_6 - s_2s_6^3 - 2s_4^2s_{12} + 2s_{20}), \quad (39)$$

$$\mathrm{PCI}(\mathrm{C_{3h}}; s_{d_{jk}}) = \tfrac{1}{2}(s_2s_3^2s_6^2 - s_2s_6^3), \quad (40)$$

$$\mathrm{PCI}(\mathrm{D}_3; s_{d_{jk}}) = \tfrac{1}{2}(s_2s_6^3 - s_2s_6^3), \quad (41)$$

$$\mathrm{PCI}(\mathrm{D_{2d}}; s_{d_{jk}}) = s_4^3s_8 - s_4^2s_{12}, \quad (42)$$

$$\mathrm{PCI}(\mathrm{D}_5; s_{d_{jk}}) = \tfrac{1}{2}(s_{10}^2 - s_{20}), \quad (43)$$

$$\mathrm{PCI}(\mathrm{T}; s_{d_{jk}}) = \tfrac{1}{2}(s_4^2s_{12} - s_4^2s_{12}), \quad (44)$$

$$\mathrm{PCI}(\mathrm{D_{3h}}; s_{d_{jk}}) = s_2s_6^3 - s_{20}, \quad (45)$$

$$\mathrm{PCI}(\mathrm{D_{5m}}; s_{d_{jk}}) = 0, \quad (46)$$

$$\mathrm{PCI}(\mathrm{T_d}; s_{d_{jk}}) = s_4^2s_{12} - s_{20}, \quad (47)$$

$$\mathrm{PCI}(\mathrm{I}; s_{d_{jk}}) = 0, \quad (48)$$

$$\mathrm{PCI}(\mathrm{S}^{[5]}; s_{d_{jk}}) = s_{20}. \quad (49)$$

Introduction of the figure-inventory $s_d = 1 + x^d$ into each of these PCIs and expansion of the resulting equation afford a generating function in which the coefficient of the x^{θ_2} term indicates the number of digraphs with θ_2 directed edges and the respective automorphism group. For example, the counting of digraphs with a T_d group is accomplished as follows:

$$\begin{aligned}\sum_{[\theta]} A_{\theta T_d} W_\theta &= \mathrm{PCI}(T_d; 1 + x^d) \\ &= (1+x^4)^2(1+x^{12}) - (1+x^{20}) \\ &= 2x^4 + x^8 + x^{12} + 2x^{16}.\end{aligned} \tag{50}$$

The results are identical with the values listed in the T_d column of table 4. They are depicted in fig. 1.

4.4. ENUMERATION BASED ON CIS

The total number (A_θ) concerning each of the weights W_θ is calculated by means of theorem 3. The cycle index (eq. (8)) for this case is obtained to be

$$\mathrm{CI}(S^{[5]}; s_{d_{jk}}) = \frac{1}{120}s_1^{20} + \frac{1}{8}s_2^{10} + \frac{1}{12}s_1^6 s_2^7 + \frac{1}{6}s_1^2 s_3^6 + \frac{1}{4}s_4^5 + \frac{1}{5}s_5^4 + \frac{1}{6}s_3^2 s_6^2, \tag{51}$$

where each variable is adopted from the $S^{[5]}(/C_{3v})$ row of table 3, and the coefficient of the variable ($\sum_{i=1}^{s} \overline{m}_{ji}$ value of each $S_j^{[5]}$) is found in the rightmost column of table 2 or at the bottom of table 3. The coefficient is positive if $S_j^{[5]}$ is a cyclic group; otherwise, it is equal to zero [19]. According to eq. (8), we have

$$\begin{aligned}\sum_{[\theta]} A_\theta W_\theta &= \mathrm{CI}(S^{[5]}; 1 + x^d) \\ &= \frac{1}{120}(1+x)^{20} + \frac{1}{8}(1+x^2)^{10} + \frac{1}{12}(1+x)^6(1+x^2)^7 \\ &\quad + \frac{1}{6}(1+x)^2(1+x^3)^6 + \frac{1}{4}(1+x^4)^5 + \frac{1}{5}(1+x^5)^4 \\ &\quad + \frac{1}{6}(1+x^2)(1+x^3)^2(1+x^6)^2 \end{aligned} \tag{52}$$

$$\begin{aligned} &= x^{20} + x^{19} + 5x^{18} + 16x^{17} + 61x^{16} + 154x^{15} + 379x^{14} + 707x^{13} \\ &\quad + 1155x^{12} + 1490x^{11} + 1670x^{10} + 1490x^9 + 1155x^8 + 707x^7 \\ &\quad + 379x^6 + 154x^5 + 61x^4 + 16x^3 + 5x^2 + x + 1.\end{aligned} \tag{53}$$

The coefficients of the resulting series (eq. (53)) are equal to the total values obtained by summing up the respective rows, as listed in the rightmost column of table 4.

By substituting 1 for x, eq. (52) (or eq. (53)) affords

$$\frac{2^{20}}{120}+\frac{2^{10}}{8}+\frac{2^6 2^7}{12}+\frac{2^2 2^6}{6}+\frac{2^5}{4}+\frac{2^4}{5}+\frac{2\times 2^2 2^2}{6}=9608,$$

which is again equal to the un-itemized value reported in ref. [4].

5. Enumeration based on elementary superposition

In the preceding section, evaluation of the $\rho_{\theta i}$ values has been accomplished by using generating functions (lemma 1). We use here an alternative method, which is based on the concept of *elementary superposition* [17,18].

In agreement with a partition $[\theta]$ (eq. (3)), we consider the symmetric group $\$^{[\theta_r]}$ whose degree is equal to θ_r ($r = 1, 2, \ldots, |\mathrm{X}|$). According to Pólya [1], the cycle index for this group is given to be

$$\mathrm{CI}(\$^{[\theta_r]}; s) = \sum_{(\nu(\theta_r))} \frac{n_{(\nu(\theta_r))}}{\theta_r!} s_1^{\nu_1(\theta_r)} s_2^{\nu_2(\theta_r)} \ldots s_{\theta_r}^{\nu_{\theta_r}(\theta_r)}, \tag{54}$$

where the cycle structure $(\nu(\theta_r)$ is represented by

$$(\nu(\theta_r)) : 1\nu_1(\theta_r) + 2\nu_2(\theta_r) + \ldots + \theta_r \nu_{\theta_r}(\theta_r) = \theta_r, \tag{55}$$

and where the coefficient of each term is represented by

$$\frac{n_{(\nu(\theta_r))}}{\theta_r!} = \frac{1}{1^{\nu_1(\theta_r)}\nu_1(\theta_r)!\, 2^{\nu_2(\theta_r)}\nu_2(\theta_r)! \ldots \theta_r^{\nu_{\theta_r}(\theta_r)}\nu_{\theta_r}(\theta_r)!}. \tag{56}$$

The direct product of the symmetric groups, i.e.

$$\mathrm{H} = \$^{[\theta_1]} \times \$^{[\theta_2]} \times \ldots \times \$^{[\theta_{|\mathrm{X}|}]}, \tag{57}$$

affords a cycle index, which is the product of the cycle indices of the factors. Hence, eq. (54) affords

$$\mathrm{CI}(\mathrm{H}; s) = \prod_{r=1}^{|\mathrm{X}|} \mathrm{CI}(\$^{[\theta_r]}; s) \tag{58}$$

$$\equiv \sum_{(\eta)} a_\eta s_1^{\eta_1} s_2^{\eta_2} \ldots s_q^{\eta_q}, \tag{59}$$

where $q = |\Delta'|$ and (η) runs over partitions represented by

$$(\eta) : 1\eta_1 + 2\eta_2 + \ldots + q\eta_q = q. \tag{60}$$

Note that the partition (η) is associated with $[\theta]$ via eqs. (58) and (59).

Since the subduced cycle indices (eq. (5)) are monomials, we can write them as follows:

$$\mathrm{ZI}(\mathrm{S}_j^{[n]}; s_{d_{jk}}) \equiv s_1^{\mu_1} s_2^{\mu_2} \ldots s_q^{\mu_q}, \tag{61}$$

where a set of $\mu_1, \mu_2, \ldots,$ and μ_q is determined to be a particular parition that depends upon eq. (5). Let $a_{\eta(\mu)}$ denote the coefficient of the monomial of eq. (59) in which (η) is identical to (μ). Then, we arrive at the concept of *elementary superposition*:*

THEOREM 6 (Theorem 18.2 of ref. [17]). (Elemenary superposition)

The $\rho_{\theta j}$ value is calculated by

$$\rho_{\theta j} = a_{\eta(\mu)}(1^{\mu_1}\, \mu_1!\; 2^{\mu_2}\, \mu_2! \ldots q^{\mu_q} \mu_q!) \tag{62}$$

$$= \mathrm{CI}(\mathrm{H}; s) * \mathrm{ZI}(\mathrm{S}_j^{[n]}; s_{d_{jk}}) \tag{63}$$

for $j = 1, 2, \ldots, s$.

The cycle structure $(\eta(\mu))$ is associated with the partition $[\theta]$ via eqs. (58) and (59). The introduction of the $\rho_{\theta j}$ values evaluated by theorem 6 into theorem 1 provides us with another tool of enumeration.

Equation (62) is converted into eq. (63) by employing the operation $(*)$ introduced by Read [3]. It should be noted that the resulting equation is a monomial, whereas the original definition of the $*$ operation is concerned with polynomials. This fact indicates that the concept of elementary superposition is an alternative foundation of the Read–Redfield superposition theorem other than the previous one from which the theorem has been derived [3]. In other words, the Read–Redfield superposition theorem can alternatively be proved by starting from the present concept of elementary superposition [17,18].

For illustrating the elementary superposition theorem (theorem 6), we re-examine the above enumeration of digraphs. For the sake of simplicity, let us consider a special case in which the partition (eq. (3)) is represented by $\theta_1 = 2$ and $\theta_2 = 18$. This means that we take account of $\mathrm{H} = \$^{[2]} \times \$^{[18]}$ and its cycle index $\mathrm{CI}(\$^{[2]}; s) \times \mathrm{CI}(\$^{[18]}; s)$.

For the SCI (s_1^{20}) of C_1, we have a combination of $s_1^2 \times s_1^{18}$, which remains effective in the cycle index of H. Since the coefficient of the term s_1^2 in $\mathrm{CI}(\$^{[2]}; s)$ is equal to $1/(1^2 2!)$ and that of the term s_1^{18} in $\mathrm{CI}(\$^{[18]}; s)$ is calculated to be

*The original proof of this theorem [21] is based on the Read–Redfield superposition theorem. However, the elementary superposition theorem can be proved directly, appendix A of ref. [18]. Thereby, it is conversely used to give an alternative proof of the Read–Redfield superposition theorem [17].

$1/(1^{18}18!)$ by means of eq. (56), the product of them is the coefficient of the s_1^{20} term in the cycle index of H. Hence, theorem 6 affords

$$\{s_1^2 \times s_1^{18}\} : \rho_{\theta C_1} = \underbrace{\frac{1}{1^2 2!}}_{(a)} \times \underbrace{\frac{1}{1^{18}18!}}_{(b)} \times \underbrace{1^{20}20!}_{(c)} = 190,$$

where the top term in braces represents such an effective combination. For memorizing this procedure, note that the part marked with (a) corresponds to the term s_1^2 of $s_1^2 \times s_1^{18}$; the part (b) to s_1^{18} of $s_1^2 \times s_1^{18}$; and part (c) to the SCI ($s_1^{20} = s_1^2 \times s_1^{18}$). Along the same line, we have the following results:

$$\{s_2 \times s_2^9\} : \rho_{\theta C_2} = \frac{1}{1^1 1!} \times \frac{1}{2^9 9!} \times 2^{10}10! = 10,$$

$$\{s_2 \times s_1^6 s_2^6, s_1^2 \times s_1^4 s_2^7\} : \rho_{\theta C_s} = \left(\frac{1}{2^1 1!} \times \frac{1}{1^6 6! 2^2 6!} + \frac{1}{1^2 2!} \times \frac{1}{1^4 4! 2^7 7!}\right) \times 1^6 6! \times 2^7 7! = 22,$$

$$\{s_1^2 \times s_3^6\} : \rho_{\theta C_3} = \frac{1}{1^2 2!} \times \frac{1}{3^6 6!} \times 1^2 2! \times 3^6 6! = 1,$$

$$\{\text{none}\} : \rho_{\theta S_4} = 0,$$

$$\{s_2 \times s_2^5 s_4^2\} : \rho_{\theta C_{2v}} = \frac{1}{1^2 2!} \times \frac{1}{2^5 5! 4^2 2!} \times 2^6 6! \times 4^2 2! = 6,$$

$$\{\text{none}\} : \rho_{\theta D_2} = 0,$$

$$\{\text{none}\} : \rho_{\theta C_5} = 0,$$

$$\{s_1^2 \times s_3^4 s_6\} : \rho_{\theta C_{3v}} = \frac{1}{1^2 2!} \times \frac{1}{3^4 4! 6^1 1!} \times 1^2 2! \times 3^4 4! \times 6^1 1! = 1,$$

$$\{s_2 \times s_3^2 s_6^2\} : \rho_{\theta C_{3h}} = \frac{1}{2^1 1!} \times \frac{1}{3^2 2! 6^2 2!} \times 2^1 1! \times 3^2 2! \times 6^2 2! = 1,$$

$$\{s_2 \times s_6^3\} : \rho_{\theta D_3} = \frac{1}{2^1 1!} \times \frac{1}{6^3 3!} \times 2^1 1! \times 6^3 3! = 1,$$

$$\{\text{none}\} : \rho_{\theta D_{2d}} = 0,$$

$$\{\text{none}\} : \rho_{\theta D_5} = 0,$$

$$\{\text{none}\} : \rho_{\theta T} = 0,$$

$$\{s_2 \times s_6^3\} : \rho_{\theta \mathrm{D}_{3h}} = \frac{1}{2^1 1!} \times \frac{1}{6^3 3!} \times 2^1 1! \times 6^3 3! = 1,$$

$$\{\text{none}\} : \rho_{\theta \mathrm{D}_{5m}} = 0,$$

$$\{\text{none}\} : \rho_{\theta \mathrm{T}_d} = 0,$$

$$\{\text{none}\} : \rho_{\theta \mathrm{I}} = 0,$$

$$\{\text{none}\} : \rho_{\theta \mathrm{S}^{[5]}} = 0.$$

These values are collected to form a row vector:

$$\mathrm{FPV} = (190, 10, 22, 1, 0, 6, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0).$$

This FPV is identical to the x^2 (or x^{18}) row of table 3. Obviously, the multiplication of the FPV by the inverse (table 2) affords the same row vector as shown in the x^2 (or x^{18}) row of table 4.

By applying the $*$ operation to the PCI (eq. (7)) and subsequently by using theorem 6, we arrive at the following theorem.

THEOREM 7 (Theorem 18.3 of ref. [17]). (Partial superposition)

The number ($A_{\theta i}$) of functions with the W_θ weight and the $\mathrm{S}_i^{[n]}$ automorphism group is represented by

$$A_{\theta i} = \mathrm{CI}(\mathrm{H}; s) * \mathrm{PCI}(\mathrm{S}_i^{[n]}; s_{d_{jk}}). \tag{64}$$

This theorem affords the $A_{\theta i}$ value with a specific $[\theta]$ and a specific $\mathrm{S}_i^{[n]}$ without utilizing generating functions.

For exemplifying the partial superposition, we re-examine the enumeration of digraphs with the $\mathrm{T_d}$ automorphism group and $\theta_1 = 2$ and $\theta_2 = 18$. Among the terms contained in the cycle index $\mathrm{CI}(\$^{[2]}; s) \times \mathrm{CI}(\$^{[18]}; s)$, we take account of $s_1^6 s_2^7$, $s_2^6 s_4^3$, $s_1^2 s_3^4 s_6$, $s_2 s_3^2 s_6^2$, $s_2 s_6^3$, $s_4^2 s_{12}$, and s_{20} because the PCI for $\mathrm{C_s}$ (eq. (33)) contains these terms. As shown above, $s_1^6 s_2^7$ are associated with effective combinations $\{s_2 \times s_1^6 s_2^6,\ s_1^2 \times s_1^4 s_2^7\}$, and the other terms can also be related to such effective combinations. Theorem 7 is applied to this case to afford

$$A_{\theta \mathrm{C}_s} = \frac{1}{6} \times \left(\left(\frac{1}{2^1 1!} \times \frac{1}{1^6 6!\, 2^6 6!} + \frac{1}{1^2 2!} \times \frac{1}{1^4 4!\, 2^7 7!} \right) \times 1^6 6! \times 2^7 7! \right)$$

$$- \frac{1}{6} \times 3 \times \left(\frac{1}{1^2 2!} \times \frac{1}{2^5 5!\, 4^2 2!} \times 2^6 6! \times 4^2 2! \right)$$

$$-\frac{1}{6}\times 3\times\left(\frac{1}{1^2 2!}\times\frac{1}{3^4 4!\, 6^1 1!}\times 1^2 2!\times 3^4 4!\times 6^1 1!\right)$$

$$-\frac{1}{6}\times\left(\frac{1}{2^1 1!}\times\frac{1}{3^2 2!\, 6^2 2!}\times 2^1 1\times 3^2 2!\times 6^2 2!\right)$$

$$+\left(\frac{1}{2^1 1!}\times\frac{1}{6^3 3!}\times 2^1 1!\times 6^3 3!\right)+0+0$$

$$=\frac{1}{6}\times(22-3\times 6-3\times 1-1+6+0+0)=1.$$

This value is equal to the one that appears in the intersection between the x^2 row and the C_s column of table 4.

The relationship between the elementary superposition and the partial superposition is verified more clearly by the following expression:

$$A_{\theta C_s}=\frac{1}{6}\times(\rho_{\theta C_s}-3\rho_{\theta C_{2v}}-3\rho_{\theta C_{3v}}-\rho_{\theta C_{3h}}+6\rho_{\theta D_{3h}}+6\rho_{\theta T_d}-6\rho_{\theta S^{[5]}})$$

$$=\frac{1}{6}\times(22-3\times 6-3\times 1-1+6+0+0)=1.$$

6. Conclusion

Enumeration of digraphs with a given automorphism group is accomplished:

(1) by using subduced cycle indices (SCIs),
(2) by using partial cycle indices (PCIs),
(3) by applying the elementary superposition to the SCIs, and
(4) by applying the partial superpositions to the PCIs.

The former two methods are based on generating functions and the latter two methods do not use such generating functions. All of these methods stem from the concept of unit subduced cycle indices (USCIs), which are derived from the subduction of coset representations.

In the present paper, we have focused attention on $S^{[5]}$. For further enumeration, we should precalculate the USCIs for the symmetric groups $S^{[n]}$ ($n \geq 6$); this task will provide us with promising results. Since the USCIs are associated with the structure of a finite group, the enumeration of digraphs (and graphs) with a given automorphism group requires knowledge on such structure, especially on the group–subgroup relationship of the group.

The methods of the USCI approach provide us with tools for itemized enumeration, which has not been accomplished by the Pólya–Redfield theorem or by the Read–Redfield theorem. Moreover, it should be emphasized that the USCI concept is a key on a fundamental level to clarify the relationship between the two before-mentioned theorems.

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